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COMPLEXES OF CATEGORIES WITH ABELIAN GROUP STRUCTURE

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The notion of a complex of categories with abelian group structure arises naturally from deriving the Hattori–Villamayor–Zelinsky sequences [2, 9, 11], and is a special case of a more general concept defined in [7]. On the other hand it is also possible to derive the Hattori–Villamayor–Zelinsky sequences by the mapping cone method of MacLane [5, 6] applied to appropriate group functors as was shown in [8] and for special cases in Hattori [3, 4]. However, these group functors are not uniquely determined. In the special cases of Hattori [3, 4] there are natural ones at hand, and in the case of a general extension of commutative rings the functors are given by a certain construction [8].

In this paper the mapping cone method and the method of [9] are compared in general and proved to be equivalent under certain commutativity conditions. In part one we show that the mapping cone method can be viewed as a special case of [9]; it corresponds to strict complexes of small categories with strict abelian group structure, and these categories can be identified with homomorphisms of abelian groups. In part two a coherence theorem for complexes of categories with abelian group structure is proved. We can change such a complex into a strict complex of small categories yielding the same cohomology sequence. In part three we prove a coherence theorem for semisimplicial complexes which applies especially to the known examples. This gives implicitly another construction of abelian group functors yielding the Hattori–Villamayor–Zelinsky sequences by the mapping cone method.

For a category with abelian group structure $\mathscr{C}$ we use the additive notation $+:\mathscr{C}\times\mathscr{C}\to\mathscr{C}$, $0:\mathscr{C}\to\mathscr{C}$, $-:\mathscr{C}\to\mathscr{C}$ for the structure functors. The structure natural transformations are always denoted by a, c, e, f, i, j and those of a homomorphism $\Gamma:\mathscr{C}\to\mathscr{D}$ by t, λ, κ [10]; they are assumed *to satisfy the coherence conditions* of [10]

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(and [9]). We call Γ, or the group structure of $\mathscr{C}$, strict, if the structure natural transformations are identities. By assumption, the morphisms in a category with abelian group structure are isomorphisms.

Section 1

Let $\mathfrak{Ab}$ denote the category of small categories with strict abelian group structure, and strict homomorphisms. Next let $\mathscr{A}b$ denote the category whose objects are the homomorphisms $f: A \to B$ of abelian groups; a morphism Γ of $f: A \to B$ to $f': A' \to B'$ in $\mathscr{A}b$ is a pair $\Gamma = (\gamma, \gamma')$ of group homomorphisms $\gamma: A \to A'$ and $\gamma': B \to B'$ with $f' \circ \gamma = \gamma' \circ f$. There exists a functor

$$\mathscr{G}: \mathscr{A}b \to \mathfrak{Ab}$$

defined as follows. For an $\mathscr{A}b$-object $f: A \to B$ define the category $\mathscr{G}_f$ by $\mathrm{Ob}(\mathscr{G}_f) = B$ and

$$\mathscr{G}_f(u, v) = \{a \in A \mid u = f(a) + v\}, \quad u, v \in B,$$

where the composition is the addition in A, cf. [1], p. 394. $\mathscr{G}_f$ is an $\mathfrak{Ab}$-object where the structure is defined by the group operations in B and A. An $\mathscr{A}b$-morphism $\Gamma = (\gamma, \gamma'): f \to f'$ yields an $\mathfrak{Ab}$-morphism $\mathscr{G}_\Gamma: \mathscr{G}_f \to \mathscr{G}_{f'}$ by $\mathscr{G}_\Gamma(u) = \gamma'(u)$ and $\mathscr{G}_\Gamma(a) = \gamma(a)$ for $u \in B, a \in A$.

Theorem 1.1. *The functor $\mathscr{G}: \mathscr{A}b \to \mathfrak{Ab}$ is an equivalence.*

Proof. First consider the functor $P: \mathfrak{Ab} \to \mathfrak{Ab}$ where $P(\mathscr{C})$, $\mathscr{C} \in \mathrm{Ob}(\mathfrak{Ab})$, is defined as follows. The objects of $P(\mathscr{C})$ are the pairs (u, a) with $u \in \mathrm{Ob}(\mathscr{C})$ and $a: u \to 0$ a $\mathscr{C}$-morphism, and a $P(\mathscr{C})$-morphism $g: (u, a) \to (v, b)$ is a $\mathscr{C}$-morphism $g: u \to v$ with $a = b \circ g$. $P(\mathscr{C})$ is an $\mathfrak{Ab}$-object where the structure is induced by that of $\mathscr{C}$, cf. [9], Proposition 2.2. For any morphism $\Gamma: \mathscr{C} \to \mathscr{D}$ in $\mathfrak{Ab}$, $P(\Gamma): P(\mathscr{C}) \to P(\mathscr{D})$ is defined by $P(\Gamma)(u, a) = (\Gamma(u), \Gamma(a))$ and $P(\Gamma)(g) = \Gamma(g)$. Then we have the commutative diagram

$$\begin{tikzcd}
P(\mathscr{C}) \arrow[r, "P(\Gamma)"] \arrow[d, "\pi"'] & P(\mathscr{D}) \arrow[d, "\pi"] \\
\mathscr{C} \arrow[r, "\Gamma"] & \mathscr{D}
\end{tikzcd}$$

where π denotes the natural projection. Now let $\bar{\mathscr{G}}: \mathfrak{Ab} \to \mathscr{A}b$ be the functor which maps $\mathscr{C}$ to the group homomorphism

$$\pi: \mathrm{Ob}(P(\mathscr{C})) \to \mathrm{Ob}(\mathscr{C})$$

and Γ to $(P(\Gamma), \Gamma)$. There is a natural isomorphism $\eta: \bar{\mathscr{G}} \circ \mathscr{G} \to \mathrm{Id}_{\mathscr{A}b}$ defined for an object $f: A \to B$ in $\mathscr{A}b$ as (η_f, id_B) where

$$\eta_f^{-1}: A \to \mathrm{Ob}(P(\mathscr{G}_f))$$

maps $a \in A$ to $(f(a), a)$. To define a natural transformation $\bar{\eta}: \mathscr{G} \circ \bar{\mathscr{G}} \to \mathrm{Id}_{\mathfrak{Ab}}$, let $\bar{\eta}_{\mathscr{C}}: \mathscr{G}_\pi \to \mathscr{C}$ be the identity on $\mathrm{Ob}(\mathscr{C})$. Any $\mathscr{G}_\pi$-morphism $\alpha: u \to v$ has the form $\alpha = (x, g)$ with $x \in \mathrm{Ob}(\mathscr{C})$ and $g: x \to 0$ a $\mathscr{C}$-morphism such that

$$u = \pi(\alpha) + v = x + v.$$

Let $\bar{\eta}_{\mathscr{C}}(\alpha)$ be the morphism $g + 0: x + v \to 0 + v$. Then $\bar{\eta}_{\mathscr{C}}: \mathscr{G}_\pi \to \mathscr{C}$ is an isomorphism in $\mathfrak{Ab}$, and the theorem is proved. $\square$

Consider now an exact sequence of complexes of abelian groups

$$0 \longrightarrow X \longrightarrow A \xrightarrow{f} B \longrightarrow Y \longrightarrow 0 \tag{1.1}$$

with $X = \ker(f)$, $Y = \mathrm{coker}(f)$, and $A_n = B_n = 0$ for $n < 0$. The mapping cone method yields an exact sequence

$$\cdots \longrightarrow H^n(X) \xrightarrow{\alpha} H^n(M(f)) \xrightarrow{\beta} H^{n-1}(Y) \xrightarrow{\gamma} H^{n+1}(X) \longrightarrow \cdots \tag{1.2}$$

where the complex $M(f)$ is defined by $M(f) = \{M_n, \partial\}$,

$$M_n = A_n \times B_{n-1}, \quad \partial(x, y) = (-\partial(x), f(x) + \partial(y)),$$

for $x \in A_n$, $y \in B_{n-1}$, and the homomorphisms are defined by

$$\alpha(\text{class of } x \in X_n) = \text{class of } (x, 0),$$

$$\beta(\text{class of } (x, y) \in M_n) = \text{class of } -\bar{y},$$

and

$$\gamma(\text{class of } \bar{y} \in Y_{n-1} \text{ with } \partial(y) = f(x)) = \text{class of } \partial(x),$$

with $\bar{y} = y \bmod \mathrm{Im}(f)$. From (1.1) we obtain a sequence

$$\mathscr{C}_0 \xrightarrow{\partial} \mathscr{C}_1 \xrightarrow{\partial} \cdots \mathscr{C}_n \xrightarrow{\partial} \mathscr{C}_{n+1} \xrightarrow{\partial} \cdots \tag{1.3}$$

of objects and morphisms in $\mathfrak{Ab}$ by $\mathscr{C}_n = \mathscr{G}(f_n)$ and $\partial = \mathscr{G}(\partial, \partial)$; then ∂^2 is equal to the constant functor 0. Let C_n denote the group of isomorphism classes of objects in $\mathscr{C}_n$ and let $F_n = \mathrm{Aut}(0_n)$. The functors ∂ induce two complexes

$$\cdots \longrightarrow C_{n-1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n+1} \longrightarrow \cdots, \tag{1.4}$$

$$\cdots \longrightarrow F_{n-1} \xrightarrow{\partial} F_n \xrightarrow{\partial} F_{n+1} \longrightarrow \cdots, \tag{1.5}$$

of abelian groups. Their cohomology groups together with certain intermediate groups, $H^n(\mathscr{C})$, constitute the cohomology sequence

$$\cdots \longrightarrow H^n(F) \xrightarrow{\alpha} H^n(\mathscr{C}) \xrightarrow{\beta} H^{n-1}(C) \xrightarrow{\gamma} H^{n+1}(F) \longrightarrow \cdots \tag{1.6}$$

derived in [9].

Theorem 1.2. *The cohomology sequences* (1.2) *and* (1.6) *are isomorphic.*

Proof. From the definition of $\mathscr{G}(f_n)$ we have $C_n = \operatorname{coker}(f_n) = Y_n$ and the complex (1.4) coincides with Y. Furthermore, we can identify X_n with F_n and X with (1.5). So we have only to construct homomorphisms

$$\theta : H^n(M(f)) \to H^n(\mathscr{C})$$

with $\theta \circ \alpha = \alpha$ and $\theta \circ \beta = \beta$. If (a, v) in $M_n = A_n \times B_{n-1}$ is a cocycle, then

$$-\partial(a) = 0 \quad \text{and} \quad f_n(a) + \partial(v) = 0.$$

Thus we have a $\mathscr{G}(f_n)$-morphism $a : \partial(-v) \to 0$, and $(-v, a)$ is an object of the category $\mathscr{P}^{n-1}$ defined in [9], p. 131. The condition $\partial(a) = 0$ says that

$$\partial(a) : \partial^2(-v) \to \partial(0)$$

equals the identity of $\partial^2(-v) = \partial(0) = 0$. Therefore we have a homomorphism $Z^n(M(f)) \to H^n(\mathscr{C})$, $(a, v) \to$ class of $(-v, a)$, which induces the desired θ as is easily seen. $\square$

The statement of Theorem 1.2 is also true if we start with an arbitrary sequence (1.3) in $\mathfrak{Ab}$ with $\partial^2 = 0$ and define (1.1) by setting $f_n : A_n \to B_n$ equal to $\mathscr{G}(\mathscr{C}_n)$.

Remark 1.3. There is another proof of Theorem 1.2 by using the concept of V–Z systems [8]. Let

$$\begin{tikzcd}
{} \arrow[rr] & & X_n \arrow[rr] \arrow[d] & & X_{n+1} \arrow[rr] \arrow[d] & & {} \\
{} \arrow[urr] & & J_n \arrow[d] \arrow[urr] & & J_{n+1} \arrow[d] \arrow[urr] & & {} \\
{} \arrow[rr] \arrow[urr] & & Y_{n-1} \arrow[rr] \arrow[urr] & & Y_n \arrow[rr] \arrow[urr] & & {}
\end{tikzcd}$$

resp.

$$\begin{tikzcd}
{} \arrow[rr] & & F_n \arrow[rr] \arrow[d] & & F_{n+1} \arrow[rr] \arrow[d] & & {} \\
{} \arrow[urr] & & P^n \arrow[d] \arrow[urr] & & P^{n+1} \arrow[d] \arrow[urr] & & {} \\
{} \arrow[rr] \arrow[urr] & & C_{n-1} \arrow[rr] \arrow[urr] & & C_n \arrow[rr] \arrow[urr] & & {}
\end{tikzcd}$$

be the V–Z system associated with (1.1), resp. (1.3). In view of [8], Proposition 2.16, we have only to show both V–Z systems are isomorphic to each other. As shown above, the complexes X and Y are identified with F and C respectively. Recall that P^n is the group of isomorphism classes of objects in $\mathscr{P}^n$, where (P,g) is an object in $\mathscr{P}^n$ if and only if $(P,g)\in B_{n-1}\times A_n$ with $\partial(P)=f_n(g)$. Two objects (P,g) and (P',g') in $\mathscr{P}^n$ are isomorphic if and only if there is an element c in A_{n-1} with $(P,g)=(P',g')+(f_{n-1}(c),\partial(c))$. Thus P^n is precisely J_n the center of the square

$$\begin{tikzcd} A_{n-1} \arrow[r, "\partial"] \arrow[d, "f_{n-1}"'] & A_n \arrow[d, "f_n"] \\ B_{n-1} \arrow[r, "\partial"] & B_n \end{tikzcd}$$

It is easy to check that the above identifications give rise to the desired isomorphisms of V–Z systems.

Section 2

Consider a sequence of homomorphisms of categories with abelian group structure

$$\mathscr{C}_0 \xrightarrow{\partial} \mathscr{C}_1 \xrightarrow{\partial} \cdots \mathscr{C}_n \xrightarrow{\partial} \mathscr{C}_{n+1} \xrightarrow{\partial} \cdots \tag{2.1}$$

and suppose we have natural transformations

$$\chi:\partial\circ\partial\to 0$$

such that for all objects P,Q in $\mathscr{C}_n$ the diagrams (D.1) and (D.2)

$$\begin{tikzcd} \partial^3(P) \arrow[r, "\chi"] \arrow[d, "\partial(\chi)"'] & 0 \arrow[d, "\mathrm{id}"] \\ \partial(0) \arrow[r, "\lambda"] & 0 \end{tikzcd} \tag{D.1}$$

$$\begin{tikzcd} \partial^2(P+Q) \arrow[rr, "\partial(t)"] \arrow[d, "\chi"'] & & \partial(\partial(P)+\partial(Q)) \arrow[d, "t"] \\ 0 & 0+0 \arrow[l, "e"'] & \partial^2(P)+\partial^2(Q) \arrow[l, "\chi+\chi"'] \end{tikzcd} \tag{D.2}$$

are commutative. As a consequence of [10], part II, $\chi:\partial^2\to 0$ is a morphism of homomorphisms in the terminology of [9]. We shall prove in this section:

Theorem 2.1. *There exists a sequence of objects and morphisms*

$$\dot{\mathscr{C}}_0 \xrightarrow{\partial} \dot{\mathscr{C}}_1 \xrightarrow{\partial} \cdots \dot{\mathscr{C}}_n \xrightarrow{\partial} \dot{\mathscr{C}}_{n+1} \xrightarrow{\partial} \cdots$$

in $\mathfrak{Ab}$ *with* $\partial^2 = 0$ *whose derived cohomology sequence is isomorphic to that of* (2.1).

The crucial point is to prove that the sequence (2.1) is coherent in a certain sense. To this end, we choose a system $(I_n)_{n\geq 0}$ of nonempty disjoint sets I_n, and arbitrary maps

$$\varepsilon : I_n \to \mathrm{Ob}(\mathscr{C}_n), \quad n \geq 0.$$

Define a system $(F(I_n))_{n\geq 0}$ of sets of words over the alphabet

$$\{(\ ,\ ), +, -, \boldsymbol{\partial}\} \cup \bigcup_n (I_n \cup \{0_n\}) \quad \text{(disjoint)} \tag{2.2}$$

inductively by:

(1) $I_n \subseteq F(I_n)$, $0_n \in F(I_n)$,

(2) $v \in F(I_n) \Rightarrow \boldsymbol{\partial}(v) \in F(I_{n+1})$,

(3) $u, v \in F(I_n) \Rightarrow (u+v),\ -v \in F(I_n)$.

$F(I_0)$ is a free group-like set [8] over I_0 and $F(I_{n+1})$ is a free group-like set over $I_{n+1} \cup \boldsymbol{\partial}(F(I_n))$. The maps $\varepsilon : I_n \to \mathrm{Ob}(\mathscr{C}_n)$ can be uniquely extended to maps of group-like sets

$$\varepsilon : F(I_n) \to \mathrm{Ob}(\mathscr{C}_n) \tag{2.3}$$

in such a way that $\varepsilon(\boldsymbol{\partial}(u)) = \partial(\varepsilon(u))$ holds for all $u \in F(I_n)$. Now let $\bar{\mathscr{C}}_n$ be the category defined by

$$\mathrm{Ob}(\bar{\mathscr{C}}_n) = F(I_n), \qquad \bar{\mathscr{C}}_n(u, v) = \mathscr{C}_n(\varepsilon(u), \varepsilon(v)),$$

$u, v \in F(I_n)$, with composition induced by that of $\mathscr{C}_n$. There are natural extensions of (2.3) to fully faithful functors

$$\varepsilon : \bar{\mathscr{C}}_n \to \mathscr{C}_n.$$

Moreover, $\mathscr{C}_n$ induces an abelian group structure on $\bar{\mathscr{C}}_n$, and ε becomes a strict homomorphism. By construction, we have a functor

$$\partial : \bar{\mathscr{C}}_n \to \bar{\mathscr{C}}_{n+1}$$

which maps $u \in \mathrm{Ob}(\bar{\mathscr{C}}_n)$ to $\boldsymbol{\partial}(u)$ and a $\bar{\mathscr{C}}_n$-morphism $g : u \to v$ to $\partial(g)$: $\boldsymbol{\partial}(u) \to \boldsymbol{\partial}(v)$. It is a homomorphism by means of the natural transformation t of $\partial : \mathscr{C}_n \to \mathscr{C}_{n+1}$, and we have the commutative diagram (2.4).

$$\begin{tikzcd}
\cdots \arrow[r] & \bar{\mathscr{C}}_{n-1} \arrow[r, "\partial"] \arrow[d, "\varepsilon"] & \bar{\mathscr{C}}_n \arrow[r, "\partial"] \arrow[d, "\varepsilon"] & \bar{\mathscr{C}}_{n+1} \arrow[r] \arrow[d, "\varepsilon"] & \cdots \\
\cdots \arrow[r] & \mathscr{C}_{n-1} \arrow[r, "\partial"] & \mathscr{C}_n \arrow[r, "\partial"] & \mathscr{C}_{n+1} \arrow[r] & \cdots
\end{tikzcd} \tag{2.4}$$

Define now subcategories $\mathscr{K}_n$ of $\bar{\mathscr{C}}_n$ with $\mathrm{Ob}(\mathscr{K}_n)=\mathrm{Ob}(\bar{\mathscr{C}}_n)$ inductively by:

(1) $a_{u,v,w}, c_{u,v}, e_v, i_v, \mathrm{id}_v \in \mathscr{K}_n$ for $u, v, w \in \mathrm{Ob}(\bar{\mathscr{C}}_n)$,

(2) $t_{u,v}: \partial(u+v) \to \partial(u)+\partial(v)$ in $\mathscr{K}_{n+1}$ for $u, v \in \mathrm{Ob}(\bar{\mathscr{C}}_n)$,

(3) $\chi_v: \partial^2(v) \to 0$ in $\mathscr{K}_{n+2}$ for $v \in \mathrm{Ob}(\bar{\mathscr{C}}_n)$,

(4) $g \in \mathscr{K}_n \Rightarrow \partial(g) \in \mathscr{K}_{n+1}$,

(5) $g, h \in \mathscr{K}_n \Rightarrow g+h, -g, g \circ h$ (if defined), $g^{-1} \in \mathscr{K}_n$.

Theorem 2.2. *The categories $\mathscr{K}_n, n \geq 0$, are atomic (i.e. for every two objects u, v in $\mathscr{K}_n$, there is at most one $\mathscr{K}_n$-morphism $u \to v$).*

Proof. By (1) and (5) above, $\mathscr{K}_n$ is a subcategory with abelian group structure and by (2) and (4), $\partial: \bar{\mathscr{C}}_n \to \bar{\mathscr{C}}_{n+1}$ can be restricted to a homomorphism $\partial: \mathscr{K}_n \to \mathscr{K}_{n+1}$. It is now convenient, using an idea of M. Laplaza, to change the 'monoidal' arrows of $\mathscr{K}_n$ into identities. For this we define the subcategories $\bar{\mathscr{K}}_n$ of $\mathscr{K}_n$ with $\mathrm{Ob}(\bar{\mathscr{K}}_n)=\mathrm{Ob}(\mathscr{K}_n)$ inductively by:

(1) $a_{u,v,w}, e_v, f_v, \mathrm{id}_v \in \bar{\mathscr{K}}_n$ for $u, v, w \in \mathrm{Ob}(\mathscr{K}_n)$,

(2) $g \in \bar{\mathscr{K}}_n \Rightarrow \partial(g) \in \bar{\mathscr{K}}_{n+1}$,

(3) $g, h \in \bar{\mathscr{K}}_n \Rightarrow g+h, -g, g \circ h$ (if defined), $g^{-1} \in \bar{\mathscr{K}}_n$.

Applying the theorem on the coherence of a, e, f, it is not difficult to see that $\bar{\mathscr{K}}_n$ is atomic. Thus we can define the factor category

$$\hat{\mathscr{K}}_n=\mathscr{K}_n/\bar{\mathscr{K}}_n,$$

cf. [8, 10]. $\mathscr{K}_n$ induces an abelian group structure on $\hat{\mathscr{K}}_n$, which is now strictly associative and unital. $\mathrm{Ob}(\hat{\mathscr{K}}_n)$ may be identified with $W_n \cup \{0_n\}$, where the system $(W_n)_{n\geq 0}$ of sets of words over (2.2) is defined by:

(1) $I_n \subseteq W_n$, $-0_n \in W_n$, $\partial(0_n) \in W_{n+1}$,

(2) $v \in W_n \Rightarrow \partial(v) \in W_{n+1}$,

(3) $u, v \in W_n \Rightarrow u+v$, $-(v) \in W_n$.

It suffices to show that $\hat{\mathscr{K}}_n$ is atomic because the projection $\mathscr{K}_n \to \hat{\mathscr{K}}_n$ is an equivalence. Since $\partial: \mathscr{K}_n \to \mathscr{K}_{n+1}$ maps $\bar{\mathscr{K}}_n$ into $\bar{\mathscr{K}}_{n+1}$ we obtain the induced sequence

$$\hat{\mathscr{K}}_0 \xrightarrow{\partial} \hat{\mathscr{K}}_1 \xrightarrow{\partial} \cdots \hat{\mathscr{K}}_n \xrightarrow{\partial} \hat{\mathscr{K}}_{n+1} \xrightarrow{\partial} \cdots.$$

For this we have the natural transformation $\chi: \partial^2 \to 0$, $\chi=\chi \bmod \bar{\mathscr{K}}_n$, and (D.1), (D.2) are commutative for the objects of $\hat{\mathscr{K}}_n$. In the following we use the $\hat{\mathscr{K}}_n$-morphisms

$$\varrho_v: -(-v) \to v, \qquad k_{u,v}: -(u+v) \to -v+(-u)$$

as defined in [10]. Let T_n denote the set of $\hat{\mathscr{K}}_n$-morphisms

$$c_{u,v},\ i_v,\ j_v,\ \varrho_v,\ k_{u,v},\ t_{x,y},\ \lambda,\ \kappa_x,\ \chi_z,\ \mathrm{id}_v$$

with $u, v \in \mathrm{Ob}(\hat{\mathscr{K}}_n)$, $x, y \in \mathrm{Ob}(\hat{\mathscr{K}}_{n-1})$, $z \in \mathrm{Ob}(\hat{\mathscr{K}}_{n-2})$ subject to the following restrictions:

(1) for $c_{u,v}$, we assume u, v are in $X_n \cup (-X_n)$, $u \neq v$, $u \neq -v$, $-u \neq v$, with $X_n = I_n \cup \partial(I_{n-1})$,

(2) for $k_{u,v}$ and $t_{x,y}$ we assume $x, y, u, v \neq 0$.

Of course $\mathscr{X}_n$ and I_n are meant to be empty for $n<0$. Note in (1) that a $\mathscr{X}_n$-morphism $c : \partial^2(z) + v \to v + \partial^2(z)$ is equal to the composition

$$\partial^2(z) + v \xrightarrow{\chi + \mathrm{id}} v \xrightarrow{\mathrm{id} + \chi^{-1}} v + \partial^2(z).$$

For $g \in \mathrm{Mor}(\mathscr{X}_n)$ define the set $E(g) \subseteq \mathrm{Mor}(\mathscr{X}_n)$ of expansions of g as in [10] by:

(1) $g, -g, -(-g), \ldots \in E(g)$,

(2) $h \in E(g) \Rightarrow h + \mathrm{id}_u,\ \mathrm{id}_u + h \in E(g)$ for $u \in \mathrm{Ob}(\mathscr{X}_n)$. Then define

$$E(T_n) = \bigcup_{g \in T_n} E(g) \quad \text{and} \quad E(T_n^{-1}) = \bigcup_{g \in T_n} E(g^{-1}). \tag{2.5}$$

Our next aim is to show, that each morphism in $\mathscr{X}_n$ can be written as a composition of elements of $E(T_n) \cup E(T_n^{-1})$.

First observe that any $\mathscr{X}_n$-morphism of the form $\partial^2(g) : \partial^2(u) \to \partial^2(v)$ with $g : u \to v$ in $\mathscr{X}_{n-2}$ may be written as

$$\partial^2(u) \xrightarrow{\chi} 0 \xrightarrow{\chi^{-1}} \partial^2(v),$$

and is clearly contained in the set $\mathscr{L}_n$ of all compositions of elements of $E(T_n) \cup E(T_n^{-1})$. From this one can deduce, cf. [10], that each morphism in $\mathscr{X}_n$ is a composition of elements of $E(\bar{T}_n) \cup E(\bar{T}_n^{-1})$ with

$$\bar{T}_n = T_n \cup \partial(\bar{T}_{n-1}),$$

$\bar{T}_{n-1}$ the set of the $\mathscr{X}_{n-1}$-morphisms $t_{x,y}, \lambda, \kappa_x, \chi_z$ in T_{n-1}. But $\partial(t_{x,y})$ and $\partial(\chi_z)$ are in $\mathscr{L}_n$ since (D.1) and (D.2) are commutative. Moreover, (D.3) and (D.4) are commutative as can be seen as follows.

$$\begin{tikzcd}
\partial^2(0) \arrow[r, "\partial(\lambda)"] \arrow[d, "\chi"'] & \partial(0) \arrow[d, "\lambda"] \\
0 \arrow[r, "\mathrm{id}"] & 0
\end{tikzcd} \tag{D.3}$$

$$\begin{tikzcd}
\partial^2(-u) \arrow[rr, "\partial(\kappa)"] \arrow[d, "\chi"'] & & \partial(-\partial(u)) \arrow[d, "\kappa"] \\
0 & -0 \arrow[l, "i"'] & -\partial^2(u) \arrow[l, "-\chi"']
\end{tikzcd} \tag{D.4}$$

We can view ∂^2 as a homomorphism where $t(\partial^2)_{u,v}$ is defined as

$$\partial^2(u+v) \xrightarrow{\partial(t)} \partial(\partial(u)) + \partial(v)) \xrightarrow{t} \partial^2(u) + \partial^2(v).$$

From this definition one can deduce

$$\lambda(\partial^2)=\lambda\circ\partial(\lambda) \quad \text{and} \quad \kappa(\partial^2)=\kappa\circ\partial(\kappa).$$

Observe now that the commutativity of (D.2) says that $\chi:\partial^2\to 0$ is a coherent natural transformation. Thus (D.3) and (D.4) must be commutative because they correspond to (D.10) and (D.11) in [10], part II. But this means that $\partial(\lambda)$ and $\partial(\kappa)$ are in $\mathscr{L}_n$ and $\mathscr{L}_n=\mathrm{Mor}(\hat{\mathscr{X}}_n)$ is proved.

Now choose on each I_n a linear ordering $<$ and extend it to a linear ordering on the disjoint union $\bigcup_n I_n$ by defining $u<v$ for $u\in I_m$, $v\in I_n$ if $m<n$. Using such an ordering we can define maps

$$rg_n:\mathrm{Ob}(\hat{\mathscr{X}}_n)\to\mathbb{N}$$

as in [10] with the following properties:

(2.6) if a $\hat{\mathscr{X}}_n$-morphism $h:u\to v$ is an expansion of an element $g\neq c_{x,y}$ of T_n, then

$$rg_n(u)\geq rg_n(v),$$

and equality holds if and only if $g=\mathrm{id}$,

(2.7) if $h:u\to v$ is an expansion of $c_{x,y}\in T_n$ then $rg_n(u)>rg_n(v)$ if and only if $rg_n(x)>rg_n(y)$,

(2.8) if

$$v\xleftarrow{h_0} u\xrightarrow{h_1} w$$

are elements of $E(T_n)$ with $rg_n(v)<rg_n(u)$ and $rg_n(w)<rg_n(u)$, then $h_1\circ h_0^{-1}$ can be written as

$$v\xrightarrow{g_0} u_1\xrightarrow{g_1}\cdots u_m\xrightarrow{g_m} w$$

with $g_\mu\in E(T_n)\cup E(T_n^{-1})$ and $rg_n(u_\mu)<rg_n(u)$, $\mu=1,\ldots,m$.

In (2.8) we can restrict our attention to the case that h_0 or h_1 is an expansion of χ because all other cases have already been considered in [10]. But this case is trivial as is easily checked.

Now let $h=h_m\circ h_{m-1}\circ\cdots\circ h_1$ be an automorphism in $\hat{\mathscr{X}}_n$ with $h_\mu:v_\mu\to v_{\mu+1}$ in $E(T_n)\cup E(T_n^{-1})$, $\mu=1,\ldots,m$. Because of $\mathrm{Aut}(u)\cong\mathrm{Aut}(0_n)$ for all $u\in\mathrm{Ob}(\hat{\mathscr{X}}_n)$ we may suppose that h is an automorphism of the neutral object 0_n. Then it is clear from (2.8) that $h=\mathrm{id}$ follows by induction on $rg_n(h,h_1,\ldots,h_m)=\max_\mu(rg_n(v_\mu))$. $\square$

Now we are ready to prove Theorem 2.1. We choose I_n and $\varepsilon:I_n\to\mathrm{Ob}(\mathscr{C}_n)$ in such a way that each object of $\mathscr{C}_n$ is isomorphic to an object in $\varepsilon(F(I_n))$. The strict homomorphism $\varepsilon:\tilde{\mathscr{C}}_n\to\mathscr{C}_n$ is then an equivalence and the commutative diagram (2.4) induces an isomorphism between the derived cohomology sequences. Knowing that $\mathscr{X}_n$ is atomic, we can proceed to

$$\hat{\mathscr{C}}_n=\tilde{\mathscr{C}}_n/\mathscr{X}_n,\quad n\geq 0,$$

which are objects in $\mathfrak{Ab}$. The homomorphism $\partial:\tilde{\mathscr{C}}_n\to\tilde{\mathscr{C}}_{n+1}$ induces a strict homomorphism $\partial : \hat{\mathscr{C}}_n \to \hat{\mathscr{C}}_{n+1}$ and we get $\partial^2 = 0$ since χ is in $\mathscr{X}_{n+2}$. We have the commutative diagram (2.9) where π denotes the projection.

$$\begin{tikzcd} \cdots \arrow[r] & \hat{\mathscr{C}}_{n-1} \arrow[r, "\partial"] & \hat{\mathscr{C}}_n \arrow[r, "\partial"] & \hat{\mathscr{C}}_{n+1} \arrow[r] & \cdots \\ \cdots \arrow[r] & \bar{\mathscr{C}}_{n-1} \arrow[r, "\partial"] \arrow[u, "\pi"] & \bar{\mathscr{C}}_n \arrow[r, "\partial"] \arrow[u, "\pi"] & \bar{\mathscr{C}}_{n+1} \arrow[r] \arrow[u, "\pi"] & \cdots \end{tikzcd} \tag{2.9}$$

But the π's are equivalences and strict homomorphisms. Thus the derived cohomology sequences are isomorphic and the theorem is proved.

Section 3

Let there be given a semisimplicial complex

$$\mathscr{C}_0 \longrightarrow \mathscr{C}_1 \rightrightarrows \mathscr{C}_2 \equiv\!\!\!\!\!> \cdots \tag{3.1}$$

of categories with abelian group structure $\mathscr{C}_n$; this means we have homomorphisms

$$d_0, d_1, \ldots, d_n : \mathscr{C}_n \to \mathscr{C}_{n+1}, \quad n \geq 0,$$

and natural transformations

$$\alpha_{i,j} : d_i \circ d_j \to d_{j+1} \circ d_i, \quad i \leq j.$$

Suppose the $\alpha = \alpha_{i,j}$ are coherent in the sense of [10] where we view the composition $d_i d_j$ as a homomorphism by $t \circ d_i(t)$. Furthermore, assume that for $i \leq j \leq k$ the diagram (3.2) is commutative.

$$\begin{tikzcd} d_i d_j d_k \arrow[r, "\alpha"] \arrow[d, "d_i(\alpha)"'] & d_{j+1} d_i d_k \arrow[r, "d_{j+1}(\alpha)"] & d_{j+1} d_{k+1} d_i \arrow[d, "\alpha"] \\ d_i d_{k+1} d_j \arrow[r, "\alpha"] & d_{k+2} d_i d_j \arrow[r, "d_{k+2}(\alpha)"] & d_{k+2} d_{j+1} d_i \end{tikzcd} \tag{3.2}$$

In the following we prove a coherence theorem for the above complex which enables us to construct a complex (2.1) satisfying (D.1) and (D.2) by the usual formula for the coboundary operator ∂.

As before, choose a system $(I_n)_{n \geq 0}$ of non-empty disjoint sets I_n and maps $\varepsilon : I_n \to \mathrm{Ob}(\mathscr{C}_n)$. Let the sets $F(I_n)$ of words over the alphabet

$$\{(\ ,\ ), +, -, \boldsymbol{d}_0, \boldsymbol{d}_1, \ldots\} \cup \bigcup_n (I_n \cup \{0_n\}) \quad \text{(disjoint)} \tag{3.3}$$

be defined by:

(1) $I_n \subsetneq F(I_n)$, $0_n \in F(I_n)$,

(2) $\upsilon \in F(I_n) \Rightarrow \boldsymbol{d}_0(\upsilon), \ldots, \boldsymbol{d}_n(\upsilon) \in F(I_{n+1})$,

(3) $u, \upsilon \in F(I_n) \Rightarrow (u+\upsilon)$, $-\upsilon \in F(I_n)$.

There are unique extensions of $\varepsilon : I_n \to \mathrm{Ob}(\mathscr{C}_n)$ to maps of group-like sets $\varepsilon : F(I_n) \to \mathrm{Ob}(\mathscr{C}_n)$ with

$$\varepsilon(\boldsymbol{d}_i(\upsilon)) = d_i(\varepsilon(\upsilon)), \quad 0 \le i \le n,$$

$\upsilon \in F(I_n)$. Define the categories $\tilde{\mathscr{C}}_n$ and homomorphisms $\varepsilon : \tilde{\mathscr{C}}_n \to \mathscr{C}_n$ and $d_i : \tilde{\mathscr{C}}_n \to \tilde{\mathscr{C}}_{n+1}$, $0 \le i \le n$, as in Section 2 so that we have a commutative diagram

$$\begin{tikzcd}
\tilde{\mathscr{C}}_n \arrow[r, "d_i"] \arrow[d, "\varepsilon"'] & \tilde{\mathscr{C}}_{n+1} \arrow[d, "\varepsilon"] \\
\mathscr{C}_n \arrow[r, "d_i"] & \mathscr{C}_{n+1}
\end{tikzcd}$$

Now define subcategories $\mathscr{X}_n$ of $\tilde{\mathscr{C}}_n$ with $\mathrm{Ob}(\mathscr{X}_n) = \mathrm{Ob}(\tilde{\mathscr{C}}_n) = F(I_n)$ inductively by:

(1) $a_{u,\upsilon,w}, c_{u,\upsilon}, e_\upsilon, i_\upsilon, \mathrm{id}_\upsilon$ are in $\mathscr{X}_n$ for $u, \upsilon, w \in \mathrm{Ob}(\tilde{\mathscr{C}}_n)$,

(2) $t_{x,y} : d_i(u+\upsilon) \to d_i(u) + d_i(\upsilon)$ is in $\mathscr{X}_{n+1}$ for $u, \upsilon \in \mathrm{Ob}(\tilde{\mathscr{C}}_n)$, $0 \le i \le n$,

(3) $\alpha : d_i d_j(\upsilon) \to d_{j+1} d_i(\upsilon)$ is in $\mathscr{X}_{n+2}$ for $\upsilon \in \mathrm{Ob}(\tilde{\mathscr{C}}_n)$, $0 \le i \le j \le n$,

(4) $g \in \mathscr{X}_n \Rightarrow d_0(g), \ldots, d_n(g) \in \mathscr{X}_{n+1}$,

(5) $g, h \in \mathscr{X}_n \Rightarrow g+h$, $-h, g \circ h$ (if defined), $h^{-1} \in \mathscr{X}_n$.

Theorem 3.1. *The categories $\mathscr{X}_n, n \ge 0$, are atomic.*

Proof. To simplify the categories we proceed to $\hat{\mathscr{X}}_n = \mathscr{X}_n / \bar{\mathscr{X}}_n$ where the atomic subcategory $\bar{\mathscr{X}}_n$ of $\mathscr{X}_n$ with $\mathrm{Ob}(\bar{\mathscr{X}}_n) = \mathrm{Ob}(\mathscr{X}_n)$ is defined by

(1) $a_{u,\upsilon,w}, e_\upsilon, f_\upsilon, \mathrm{id}_\upsilon \in \bar{\mathscr{X}}_n$ for $u, \upsilon, w \in \mathrm{Ob}(\mathscr{X}_n)$,

(2) $g \in \bar{\mathscr{X}}_n \Rightarrow d_0(g), \ldots, d_n(g) \in \bar{\mathscr{X}}_{n+1}$,

(3) $g, h \in \bar{\mathscr{X}}_n \Rightarrow g+h$, $-h, g \circ h$ (if defined), $h^{-1} \in \bar{\mathscr{X}}_n$.

$\hat{\mathscr{X}}_n = \mathscr{X}_n / \bar{\mathscr{X}}_n$ is a category with abelian group structure where the product is now strictly associative and unital. $\mathrm{Ob}(\hat{\mathscr{X}}_n)$ can be identified with $W_n \cup \{0_n\}$ where the sets $W_n, n \ge 0$, of words over the alphabet (3.3) are defined by:

(1) $I_n \subsetneq W_n$, $-0_n \in W_n$, $\boldsymbol{d}_0(0_n), \ldots, \boldsymbol{d}_n(0_n) \in W_{n+1}$,

(2) $\upsilon \in W_n \Rightarrow \boldsymbol{d}_0(\upsilon), \ldots, \boldsymbol{d}_n(\upsilon) \in W_{n+1}$,

(3) $u, \upsilon \in W_n \Rightarrow u + \upsilon$, $-(\upsilon) \in W_n$.

The homomorphisms $d_i : \mathscr{X}_n \to \mathscr{X}_{n+1}$, $0 \le i \le n$, induce homomorphisms $d_i : \hat{\mathscr{X}}_n \to \hat{\mathscr{X}}_{n+1}$ and the $\alpha_{i,j}$ of the original complex define natural transformations $\alpha_{i,j} : d_i d_j \to d_{j+1} d_i$, $i \le j$, for the $d_i : \hat{\mathscr{X}}_n \to \hat{\mathscr{X}}_{n+1}$. Now let $\Delta_n, n \ge 0$, be the set of $\hat{\mathscr{X}}_n$-morphisms

$$c_{u,\upsilon},\ i_\upsilon,\ j_\upsilon,\ \varrho_\upsilon,\ k_{u,\upsilon},\ \mathrm{id}_\upsilon,$$

$u, \upsilon \in \mathrm{Ob}(\hat{\mathscr{X}}_n)$ with the following restrictions:

(1) for $c_{u,v}$, u, v are in $X_n \cup (-X_n)$, $u \neq v$, $u \neq -v$, $-u \neq v$, where the sets X_n are the smallest subsets of $\mathrm{Ob}(\mathscr{X}_n)$ such that $I_n \subseteq X_n$ and $v \in X_n \Rightarrow d_0(v), \ldots, d_n(v) \in X_{n-1}$,

(2) for $k_{u,v}$, u, v are not equal to 0.

Next define the subsets $\tilde{\Delta}_n$ of $\mathrm{Mor}(\mathscr{X}_n)$ inductively by:

(1) $t : d_i(u+v) \to d_i(u) + d_i(v)$, $\lambda : d_i(0) \to 0$, and $\kappa : d_i(-v) \to -d_i(v)$ are in $\tilde{\Delta}_{n+1}$ for $u, v \in \mathrm{Ob}(\mathscr{X}_n)$, $0 \leq i \leq n$, where $u, v \neq 0$ for $t_{u,v}$,

(2) $\alpha : d_i d_j(v) \to d_{j+1} d_i(v)$ is in $\tilde{\Delta}_{n+2}$ for $u, v \in \mathrm{Ob}(\mathscr{X}_n)$, $i \leq j$,

(3) $g \in \tilde{\Delta}_n \Rightarrow d_0(g), \ldots, d_n(g) \in \tilde{\Delta}_{n+1}$.

Note that $\tilde{\Delta}_0$ is empty. Let

$$T_n = \Delta_n \cup \tilde{\Delta}_n, \quad n \geq 0,$$

and define $E(T_n)$ and $E(T_n^{-1})$ as in (2.5). Then every $\mathscr{X}_n$-morphism is a composition of elements of $E(T_n) \cup E(T_n^{-1})$. This can be seen by the same method as before. Now define maps

$$rg_n : \mathrm{Ob}(\mathscr{X}_n) \to \mathbb{N}$$

with the properties (2.6)–(2.8). Concerning (2.8), the only new diagram is (3.2); all other diagrams have already been considered in [10]. Then the same induction method clearly yields $h = \mathrm{id}$ for all automorphisms in $\mathscr{X}_n$. □

From the semisimplicial complex (3.1) we can form a complex (2.1) defining $\partial : \mathscr{C}_n \to \mathscr{C}_{n+1}$ by the usual formula

$$\partial(P) = (\cdots((d_0(P) + (-d_1(P))) + d_2(P)) + \cdots) + (\pm d_n(P)).$$

The natural transformations $\chi : \partial^2 \to 0$ can be constructed from the $\alpha_{i,j}$ and clearly the commutativity conditions will be satisfied by Theorem 3.1. Observe that the same construction for the $d_i : \bar{\mathscr{C}}_n \to \bar{\mathscr{C}}_{n+1}$ defined above yields a complex

$$\cdots \longrightarrow \bar{\mathscr{C}}_{n-1} \xrightarrow{\partial} \bar{\mathscr{C}}_n \xrightarrow{\partial} \bar{\mathscr{C}}_{n+1} \longrightarrow \cdots \tag{3.4}$$

and a commutative diagram (2.4). Assuming then that the $\varepsilon : \bar{\mathscr{C}}_n \to \mathscr{C}_n$ are equivalences, the derived cohomology sequences are isomorphic. Furthermore, it is now possible by Theorem 3.1 to define $\hat{\mathscr{C}}_n = \bar{\mathscr{C}}_n / \mathscr{X}_n$. Then we get the semisimplicial complex

$$\hat{\mathscr{C}}_0 \longrightarrow \hat{\mathscr{C}}_1 \rightrightarrows \hat{\mathscr{C}}_2 \equiv \cdots$$

with $d_i d_j$ equal to $d_{j+1} d_i$, $i \leq j$, since the $\alpha_{i,j}$ are in $\mathscr{X}_n$. The derived cohomology sequence is then isomorphic to that of (3.4) via the projections $\pi : \bar{\mathscr{C}}_n \to \hat{\mathscr{C}}_n$.

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